Wavelets and Differential Equations–A short review

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In this paper some computational aspects of wavelets and various wavelet methods have been reveiwed. The description of existing numerical methods and several useful aspects of wavelets to solve the differential equations is discussed. Finally, the formulation of some wavelet based numerical methods for one dimensional problem is also discussed given briefly.

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I. INTRODUCTION:

The applications of wavelet theory in numerical methods for solving differential equations are roughly 20 years old. In the early nineties people were very optimistic because it seemed that many nice properties of wavelets would automatically leads to efficient numerical method for differential equations. The reason for this optimism was the fact that many nonlinear partial differential equations (PDEs) have solution containing local phenomena (e.g. formation of shock, hurricanes) and interactions between several scales (e.g. turbulence particularly atmospheric turbulence because there is motion on a continuous range of length scales). Such solutions can be well represented in wavelet bases because of its nice properties few of them like compact support (locality in space) and vanishing moment (locality in scale). Furthermore, this early optimism has been already honored by many authors [1–6] working in this area since then. Nevertheless, there often remains a large gap between a theoretical wavelet paper and the needs of an applied mathematician. This paper is an attempt to bridge this gap by providing a short review on wavelet based numerical methods for differential equations.

Most common numerical methods used for numerical solution of physical problems (mostly leads to partial differential equation) fall in to following classes.

- ***Finite difference methods (FDM)***
  The different unknowns are defined by their values on discrete (finite) grid and differential operators are replaced by difference operators using neighboring points. See [7–9] for details.

- ***Finite volume methods (FVM)***
  Similar to the finite difference method, values are calculated at discrete places on a meshed geometry. "Finite volume" refers to the small volume surrounding each node point on a mesh. See [10] for details.

- ***Finite elements methods (FEM)***
  The unknown solution is approximated by a linear combination of a set of linearly independent test functions, which are piecewise continuous and non vanishing only on the finite number of elements in the domain. Examples of methods that use higher degree piecewise polynomial basis functions are the hp-FEM. See [11] for details.

- ***Spectral methods***
  Utilizing basis functions which are infinitely differentiable and non vanishing on the entire domain (global support). See [12, 13] for details.

- ***Wavelet methods***
  Utilizing basis functions which are differentiable (according to the requirement) and non vanishing on the compact support. See [14] for details.

Moreover, FDM and FVM are an approximation to the differential equation while other methods are an approximation to its solution. As we noted earlier, spectral bases are infinitely differentiable, but have global support. On the

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other hand, bases functions used in finite difference or finite element methods have small compact support but poor continuity properties. Conclusively, spectral method have good accuracy, but poor spatial localization, while FDM, FVM and FEM have good spatial localization but poor accuracy. Wavelet based numerical methods seem to combine the advantage (spectral accuracy as well good localization) of all the methods using wavelet bases. Schematically, common wavelet based numerical methods for differential equations can be separated into the following categories:

- ***Category* 1*: Methods based on scaling function expansion***
  The unknown solution is expanded in scaling function at some chosen level J and differential equation is solved using a Galerkin approach. This approach can't exploit wavelet compression hence methods in this category are not adaptive [15–18].

- ***Category* 2*: Methods based on wavelet expansion***
  The differential equation is solved using a Galerkin approach as in the first category. However, the unknown solution is expressed in terms of wavelets rather than scaling function. Therefore, this approach can exploit wavelet compression; either to the solution [19], the differential operator [20], or both [21–23].

- ***Category* 3*: Wavelets and finite differences***
  Here wavelets are used to derive adaptive finite difference methods. Instead of expanding the solution in terms of scaling function or wavelet expansion, the wavelet transform is used to determine where the finite difference grid must be refined or coarsened [5, 6, 24–26].

The detailed explanation of these wavelet based numerical methods will be given in subsequent sections.

II. SOME COMPUTATIONAL ASPECTS OF WAVELETS

A. Wavelets

A wavelet is a mathematical function used to divide a given function or continuous-time signal into different scale components. The word wavelet is due to Morlet and Grossmann in the early 1980s. They used the French word ondelette, meaning "small wave". Soon it was transferred to English by translating "onde" into "wave", giving "wavelet". The study of wavelets has attained the present growth due to mathematical analysis of wavelets by Stromberg [27], Grossmann and Morlet [28] and Meyer [29]. The concept of Multiresolution Analysis (MRA) was introduced by S. Mallat [30] and Y. Meyer [29]. The first orthonormal bases of compactly supported wavelets are introduced by I. Daubechies in 1988 [31]. A review of the basic properties of the wavelets and the decomposition and the reconstruction of functions in terms of the wavelet bases is given by Strang [32] and the details of the mathematical analysis of wavelets as approximation of functions in $L_2(\mathbb{R})$ are described in [33].

Given a basis $\{f_k\}_{k\in I}$ in a Hilbert space $\mathcal{H}$, every $f \in \mathcal{H}$ can be uniquely represented as

$$f = \sum_{k\in I} c_k(f) f_k. \tag{1}$$

A frame is also a set $\{f_k\}_{k\in I}$ in $\mathcal{H}$ which allows every f to be written like (1) but it may be linearly dependent (if frame is linearly independent set for $L_2(\mathbb{R})$ then frame gives Riesz basis for $L_2(\mathbb{R})$). Thus, one may get redundant representation. More precisely, a family $\{f_k\}_{k\in I}$ in a Hilbert space $\mathbb{H}$ is a frame for $\mathbb{H}$ if $\exists$ two constant $m > 0$, $M < \infty$ such that

$$m||f||^2 \leq \sum_{k\in I} | < f, f_k > |^2 \leq M||f||^2, \forall f \in \mathbb{H}. \tag{2}$$

Moreover, for every frame $\exists$ a dual frame $\{\tilde{f}_k\}_{k\in I}$ such that

$$f = \sum_{k\in I} < f, \tilde{f}_k > f_k, \forall f \in \mathbb{H}. \tag{3}$$

Therefore, bases are optimal for fast data processing, whereas the redundancy inherent in frames increase flexibility and robustness to the noise, but usually at the price of high computational cost.

Multiresolution analysis (MRA) [30] is the theory that was used by I. Daubechies to show that for any non negative integer n there exists an orthogonal wavelet with compact supports such that all the derivatives up to order n exist and characterized by the following axioms:

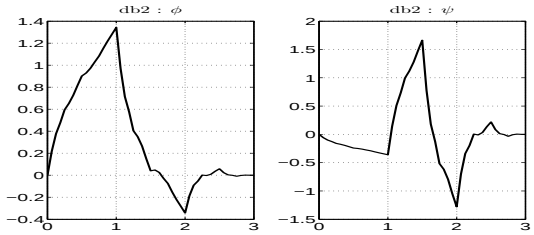

FIG. 1:

- $\mathcal{V}^j \subset \mathcal{V}^{j+1}$ (subspaces are nested),
- $f \in \mathcal{V}^j$ iff $f(2(.)) \in \mathcal{V}^{j+1}$ for all $j \in \mathbb{Z}$ (invariance to dialation),
- $\overline{\bigcup_{j\geq 0} \mathcal{V}^j} = L_2(\mathbb{R})$,
- $\{\phi(x-k)|k \in \mathbb{Z}\}$ is an orthonormal basis for $\mathcal{V}^0$ (invariance to translation).

We define $\mathcal{W}^j$ to be the orthogonal complement of $\mathcal{V}^j$ in $\mathcal{V}^{j+1}$, i.e. $\mathcal{V}^j \perp \mathcal{W}^j$ and

$$\mathcal{V}^{j+1} = \mathcal{V}^j + \mathcal{W}^j. \tag{4}$$

There exists a function, which is called a **scaling function** $\phi(x) \in \mathcal{V}^0$, such that the sequence $\phi_k^j(x) = 2^{j/2}\phi(2^j x - k)_{k\in\mathbb{Z}}$ is an orthonormal basis for $\mathcal{V}^j$ and similarly there exist a function $\psi(x) \in \mathcal{W}^0$ (which is called **mother wavelet**) such that $\psi_k^j(x) = 2^{j/2}\psi(2^j x - k)_{k\in\mathbb{Z}}$ is an orthonormal basis for $\mathcal{W}^j$. Since $\phi_0^0(x) = \phi(x) \in \mathcal{V}^0 \subset \mathcal{V}^1$, so

$$\phi(x) = \sum_{k=-\infty}^{\infty} h_k \phi_k^1(x). \tag{5}$$

Eq. (5) is called **dialation equation** (two scale relation for scaling function) and for Daubechies compactly supported scaling function only finitely many $h_k, k = 0, 1, \cdots D-1$ will be nonzero. Where D is even positive integer called the **wavelet genus** and $h_0, h_1, \cdots, h_{D-1}$ are called **low pass filter coefficients**. Similarly, Daubechies compactly supported wavelet $\psi(x) \in \mathcal{W}^0 \subset \mathcal{V}^1$, therefore

$$\psi(x) = \sum_{k=0}^{D-1} g_k \phi_k^1(x). \tag{6}$$

Eq. (6) is called **wavelet equation** (two scale relation for wavelet function) and $g_0, g_1, \cdots, g_{D-1}$ are called **high pass filter coefficients**. These filter coefficients are connected by the relation $g_k = (-1)^k h_{D-1-k}, k = 0, 1, \cdots, D-1$.

The MATLAB function $[h_k,\ g_k]$=**wfilters('dbM','r')** computes these filter coefficients, where $M = \frac{D}{2}$, see [34] for details (e.g.: For Haar wavelet $M = 1, D = 2, h_k = [\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}]$ and $g_k = [\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}]$).

One should notice that there is no closed form analytic (explicit) formula for Daubechies scaling function ($\phi(x)$) and wavelet function ($\psi(x)$) (except Haar scaling function ($\phi(x) = 1$ if $x \in [0, 1]$, $\phi(x) = 0$ otherwise) and Haar wavelet function ($\psi(x) = 1$ if $x \in [0, .5]$, $\psi(x) = -1$ if $x \in [.5, 1]$, $\psi(x) = 0$ otherwise)) and it's value can be computed only at dyadic points using the cascade algorithm [34, 35].

The MATLAB function
$[\phi_i,\ \psi_i,\ x_i]$=**wavefun('dbM',iter)** computes the value of $\phi(x)$ and $\psi(x)$ at the grid $x_i = [0, \frac{1}{2^{iter}}, \cdots, D-1]$, see [35] for details. The functions $\phi(x)$ and $\psi(x)$ are plotted in Fig. 1 for 'db2' and iter=4.

In any wavelet bases or frames, a number of additional properties are desirable, such as smoothness, orthogonality, compact (local) support, Riesz stability, vanishing moments. We will justify few of them.

- **Smoothness**: Smoothness is needed to approximate smooth data.

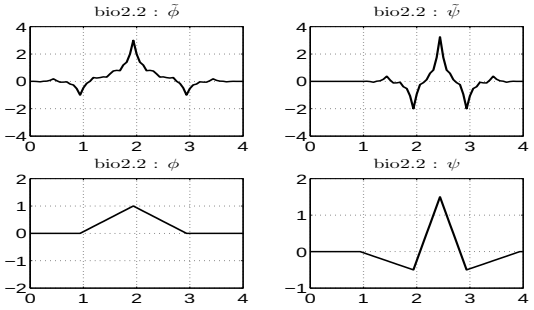

FIG. 2:

- **Orthogonality**: For numerical computation orthogonality leads to fast algorithm. However, orthogonality is difficult to achieve (like there is no symmetric orthogonal wavelet ψ with compact support) otherwise leads to the loss of other useful properties. Therefore, in numerical applications orthogonality is relaxed to either semiorthogonality of biorthogonality. Important references for biorthogonal wavelets are [34, 36]. The orthogonal wavelet of Daubechies involve a scaling function $\phi(x)$ and $\psi(x)$ as discussed in Sec. II A. In case of biorthogonal wavelet every multiresolution analysis is accompanied by dual multiresolution analysis consisting of nested space $\tilde{\mathcal{V}}^j$ with bases given by dual scaling function $\tilde{\phi}^j_k(x)$ which are biorthogonal to scaling function

$$< \phi^j_k, \tilde{\phi}^j_{k'} >= \delta_{k,k'}, \text{ for } k, k^{'} \in I.$$

  Moreover the pair $(\psi^j_k, \tilde{\psi}^{j^{'}}_{k'})$ satisfy the following biorthogonality property

$$< \psi^j_k, \tilde{\psi}^{j^{'}}_{k'} >= \delta_{j,j'}\delta_{k,k'}, \text{ for } j, j^{'}, k, k^{'} \in I.$$

The MATLAB function
[$\tilde{\phi}_i$,$\tilde{\psi}_i$,ϕ_i,ψ_i,x_i]=**wavefun('bior2.2',4)** computes the value of $\tilde{\phi}(x)$, $\tilde{\psi}(x)$, $\phi(x)$ and $\psi(x)$ at the grid $x_i = [0, \frac{1}{2^{iter}}, \cdots, D-1]$, see [34] for details. The functions $\tilde{\phi}(x)$, $\tilde{\psi}(x)$, $\phi(x)$ and $\psi(x)$ are plotted in Fig. 2 for 'bio2.2' (both wavelet and dual wavelet have 2 vanishing moments) and iter=4.

If $\phi^j_k = \tilde{\phi}^j_k, \forall j, k \implies$ orthogonal scaling function.
If $\mathcal{V}^j = \tilde{\mathcal{V}}^j, \forall j \implies$ semiorthogonal scaling function.

- **Compact support**: It also leads to the fast algorithm for numerical computation.
- **Vanishing moment**: The wavelet is said to have $M(M \in \mathbb{N})$ vanishing moment if it verifies the following condition

$$\int_{\mathbb{R}} x^n \psi(x) = 0, \text{for}, n = 0, 1, \cdots, M-1,$$

  (equivalently scaling function can represents polynomials of degree up to $M-1$ exactly). This property improves the efficiency of $\psi(x)$ at detecting singularities in the signal (therefore, wavelet bases are suitable for representing piecewise smooth function).

1. Univariate wavelet

A "Wavelet System" consists of the scaling function $\phi(x)$ and the wavelet function $\psi(x)$. In literature, several wavelets with different properties have been derived. Few examples of them are given below.

- Haar wavelet ([37]),
- Daubechies wavelets with different compact, supports ([31]),
- Coiflet (Beylkin *et al.* [20], Daubechies [34]),

- Block spline semi-orthogonal wavelets (Chui and Wang [38]),
- Battle-Lemarie's wavelets (Battle [39]),
- Biorthogonal wavelets of Cohen *et al.* ([36]),
- Shannon's wavelet and Meyer's wavelet ([29]).

2. Multivariate wavelet

The simplest way to obtain multivariate wavelets is to employ anisotropic or isotropic tensor products: (MRA-d) Here, the multivariate wavelets are defined by

$$\begin{aligned}\psi_l^j(x) &:= \psi_{l_1}^{j_1}(x_1).\cdots.\psi_{l_d}^{j_d}(x_d),\\ j &:= (j_1,\cdots,j_d) \quad x,l \text{ analogous.}\end{aligned}$$

(MRA) Here, anisotropy is avoided by setting $j_1 = j_2 \cdots j_d = j$. The scaling functions are simply the tensor products of the univariate scaling functions. A two-dimensional MRA can be constructed from the following decomposition:

$$\begin{aligned}\mathcal{V}^{\mathbf{j}} = \mathcal{V}^j \otimes \mathcal{V}^j &= (\mathcal{V}^{j-1} \oplus W^{j-1}) \otimes (V^{j-1} \oplus W^{j-1})\\ &= (W^{j-1} \otimes \mathcal{W}^{j-1}) \oplus (\mathcal{W}^{j-1} \otimes \mathcal{V}^{j-1})\\ &\oplus (\mathcal{V}^{j-1} \otimes \mathcal{W}^{j-1}) \oplus (\mathcal{V}^{j-1} \otimes \mathcal{V}^{j-1})\\ &= \mathcal{W}^{\mathbf{j-1}} \oplus \mathcal{V}^{\mathbf{j-1}}.\end{aligned}$$

Then we have $\mathcal{V}^{\mathbf{J}} = \mathcal{W}^{\mathbf{J-1}} \oplus \cdots \oplus \mathcal{W}^{\mathbf{0}} \oplus \mathcal{V}^{\mathbf{0}}$ and wavelet basis is given by

$$\begin{aligned}&\{\psi_k^j \otimes \psi_l^j, \psi_k^j \otimes \phi_l^j, \phi_k^j \otimes \psi_l^j\}_{k,l\in\mathbb{Z},0\le j\le J-1}\\ &\cup \{\phi_k^0 \otimes \phi_l^0\}_{k,l\in\mathbb{Z}}.\end{aligned}$$

B. Periodized Wavelets

Most of the wavelet algorithms can handle periodic boundary conditions easily. However, different possibilities of dealing general boundary conditions have been studied. The few of them are as follows.

- One approach is to use wavelets specified on an interval where wavelets are constructed satisfying certain boundary conditions. To achieve wavelet approximation on a bounded interval is to keep all Daubechies's wavelets whose supports are totally inside the interval, while modifying those wavelets intersecting the boundary by an orthonormalization [40–43] (semiorthogonalization in [44, 45]). The disadvantages of this approach are inconvenience of implementation and wavelet dependence on boundary conditions.
- The variational approach suggested by Glowinski et al. [1] is not applicable for some nonlinear problems, furthermore, it is impractical for higher dimensions.
- The use of antiderivatives of wavelets as trial functions in [46]. In this way singularities in the wavelets are smoothened and the boundary conditions can be treated more easily.
- A more satisfactory approach is to use **second generation wavelets** introduced in [47]. In Sec. II A wavelet function $\psi_k^j(x)$ are traditionally defined as the dyadic translates and dilates of one particular $L_2(\mathbb{R})$ function, the mother wavelet $\psi(x)$: $\psi_k^j(x) = \psi(2^j x - j)$. These wavelets are called **first generation wavelets**. Second generation wavelets are more general, where wavelets are not necessarily dilates and translates of single function but have all nice properties of first generation wavelets, which can be used for general boundary conditions as well complex geometry [6].

We will discuss wavelet based numerical methods for numerical example with periodic boundary conditions, therefore, we explain periodized wavelets in detail. As pointed out by Y. Meyer (1990) the complete toll box built in $L_2(\mathbb{R})$ can be used in the periodic case $L_2([0,p])$ by introducing a standard periodization technique. This technique consists

at each scale in folding, around the integer values, the wavelet $\psi_k^j(x)$ and the scaling functions $\phi_k^j(x)$ centered in $[0, p]$. Let $\phi(x) \in L_2(\mathbb{R})$ and $\psi(x) \in L_2(\mathbb{R})$ be the scaling and wavelet function from a multiresolution analysis as defined in Sec. II A. For any $j, l \in \mathbb{Z}$ and $x \in \mathbb{R}$, we define the p-periodic scaling function

$$\tilde{\phi}_l^{j,p}(x) = \sum_{n=-\infty}^{\infty} \phi_l^j(x + pn) = 2^{j/2} \sum_{n=-\infty}^{\infty} \phi(2^j(x + pn) - l), \tag{7}$$

and the p-periodic wavelet

$$\tilde{\psi}_l^{j,p}(x) = \sum_{n=-\infty}^{\infty} \psi_l^j(x + pn) = 2^{j/2} \sum_{n=-\infty}^{\infty} \psi(2^j(x + pn) - l). \tag{8}$$

The p periodicity can be verified as follows

$$\begin{aligned}\tilde{\phi}_l^{j,p}(x + p) = \sum_{n=-\infty}^{\infty} \phi_l^j(x + pn + p) &= \sum_{m=-\infty}^{\infty} \phi_{j,l}(x + pm) \\ &= \tilde{\phi_l^{j,p}}(x),\end{aligned}$$

and similarly $\tilde{\psi}_l^{j,p}(x + p) = \tilde{\psi}_l^{j,p}(x)$. Where $\tilde{\phi}_l^{j,p}(x)$ and $\tilde{\psi}_l^{j,p}(x)$ generates the spaces $\mathcal{V}^{j,p}$ and $\mathcal{W}^{j,p}$ respectively. For notational convenience $\tilde{\phi}_l^{j,1}(x) = \tilde{\phi}_l^j(x)$, $\tilde{\psi}_l^{j,1}(x) = \tilde{\psi}_l^j(x)$.

C. Projection onto space $\mathcal{V}^j$

Let $P_{\mathcal{V}^j} f$ be the projection of a function f on $\mathcal{V}^j$ and

$$P_{\mathcal{V}^j} f(x) = \sum_{k=-\infty}^{\infty} c_k^j \phi_k^j(x), \quad x \in \mathbb{R}. \tag{9}$$

Two natural ways of representing f on $\mathcal{V}^j$ (equivalently going to wavelet space from physical space) are orthogonal projection and interpolation.

1. Orthogonal projection

The orthogonality of the basis suggest the orthogonal projection. In this case, the expansion coefficients are defined as

$$c_k^j = \int_{-\infty}^{\infty} f(x)\phi_k^j(x)dx.$$

This integral can be approximated by a quadrature method.

2. Interpolation

Using interpolation is also a popular choice for projecting f on $\mathcal{V}^j$. To explain the idea consider the projection of f on periodic space $\mathcal{V}^{j,1}$ such that f coincide at node points of level j, where c_k^j are the expansion coefficients.

$$f(l/2^j) = \sum_{k=0}^{2^j-1} c_k^j \tilde{\phi}_k^j(l/2^j), \quad l = 0, \cdots, 2^j - 1.$$

This can be rewritten as

$$f(l/2^j) = \sum_{k=0}^{2^j-1} c_k^j \tilde{\phi}_{k-l}^j(l/2^j), \quad l = 0, \cdots, 2^j - 1.$$

Therefore calculating the coefficients c_k^j reduces to solving a matrix equation

$$F^j = T^j c^j,$$

where F^j is the vector of components $f_l^j = f(l/2^j)$ for $l = 0, \cdots, 2^j - 1, c^j$ is the vector of coefficients c_k^j for $k = 0, \cdots, 2^j - 1$, T^j is the circulant matrix of size $N = 2^j$.
cost : If simple Gaussian elimination is used to solve this system, then the cost of finding the c_k^j is heavy: $O(n^3)$ operations, where n is the order of the matrix. Significantly better performance can be achieved by use of sparse matrix routines. However, because this system is circulant, using FFT, the solution can be found in $O(n \log_2 n)$ operations [48].

D. Wavelet transform (WT)

Wavelet transforms have advantages over traditional Fourier transforms for representing functions that have discontinuities and sharp peaks, and for accurately deconstructing and reconstructing finite, non-periodic and/or non-stationary signals. Wavelet transforms are classified into **discrete wavelet transform** (DWT) and **continuous wavelet transform** (CWT). The continuous wavelet transform $W_\psi f$ of $f \in L_2(\mathbb{R})$ with respect to ψ is defined as

$$(W_\psi)(b, a) = |a|^{-\frac{1}{2}} \int_{-\infty}^{\infty} f(x)\overline{\psi(\frac{x-b}{a})}.$$

In order to reconstruct f from $W_\psi f$, we need to know the constant

$$C_\psi = \int_{-\infty}^{\infty} \frac{|\widehat{\psi}(w)|^2}{|w|} dw < \infty.$$

The finiteness of this constant (**admissibility condition**) restrict the class of $L_2(\mathbb{R})$ functions that can be used as wavelets. Which implies

$$\int_{-\infty}^{\infty} \psi(x) = 0, \text{see [33] for details.}$$

With the constant C_ψ, we have the following reconstruction formula

$$f(x) = \frac{1}{C_\psi} \int_{\mathbb{R}^2} \int W_\psi(b, a)\overline{\psi(\frac{x-b}{a})} \frac{dadb}{a^2}, f \in L_2(\mathbb{R}).$$

Notice that the possibility of reconstruction is guaranteed by the admissibility condition. Now, in practice, numerical implementation requires the CWT to be discretized.

$$f(x) = \sum_{j,k \in \mathbb{Z}} (W_\psi f)(b_k^j, a^j)\tilde{\psi}_k^j(x), \tag{10}$$

where $a^j = \frac{1}{2^j}$ is called the binary dilation (or dyadic dilation) and $b_k^j = \frac{k}{2^j}$ is the binary or dyadic position. However, this procedure leads to the frames not to the bases. For bases, one needs another approach of DWT based on the notion of multiresolution analysis.

The orthogonal projection $P_{\mathcal{V}^j}$ is given by Eq. (9). Since space $\mathcal{V}^J$ is decomposed into wavelet space $\mathcal{V}^{J_0} + \mathcal{W}^{J_0} + \mathcal{W}^{J_0-1} + \cdots + \mathcal{W}^{J-1}$ using the relation (4), we obtain wavelet series

$$P_{\mathcal{V}^J} f(x) = \sum_{k=-\infty}^{\infty} c_k^{J_0} \phi_k^{J_0}(x) + \sum_{j=J_0}^{J-1} \sum_{k=-\infty}^{\infty} d_k^j \psi_{j,k}(x), \tag{11}$$

where J_0 (coarsest level of approximation) satisfy $0 \leq J_0 \leq J$. The $c_k^{J_0}$ (scaling coefficients) and d_k^j (wavelet coefficients) for $j = J_0, \cdots, J-1$ are given by

$$c_k^{J_0} = \int_{-\infty}^{\infty} f(x)\phi_k^{J_0}(x)dx, \tag{12}$$

$$d_k^j = \int_{-\infty}^{\infty} f(x)\psi_k^j(x)dx. \tag{13}$$

The orthonormality properties of the scaling and wavelet functions arising from a multiresolution of $L_2(\mathbb{R})$ lead to simple relations connecting the scaling coefficients and the wavelet coefficients of different levels. Mallat [49] exploited these relations to develop a fast algorithm which transforms the coefficients from one level of resolution j to the next coarse level $j-1$. This yields a fast and accurate algorithm denoted by pyramid algorithm. For Daubechies compactly supported wavelet system of genus D, using the relation (5)

$$\begin{aligned}\phi_l^{j-1} = 2^{\frac{(j-1)}{2}}\phi(2^{j-1}x - l) &= 2^{\frac{j}{2}}\sum_{k=0}^{D-1} h_k\phi(2^j x - 2l - k) \\ &= \sum_{k=0}^{D-1} h_k\phi_{2l+k}^j(x).\end{aligned}$$

Similarly, $\psi_l^{j-1}(x) = \sum_{k=0}^{D-1} g_k\phi_{2l+k}^j(x)$ using (6). Using these results in the definitions of the scaling and wavelet coefficients

$$c_l^{j-1} = \sum_{k=0}^{D-1} h_k c_{2l+k}^j, \tag{14}$$

$$d_l^{j-1} = \sum_{k=0}^{D-1} g_k c_{2l+k}^j. \tag{15}$$

Applying the Eqs. (14) and (15) recursively for $j = J, J-1, \cdots, J_0+1$, starting with the initial sequence $c_l^J, l \in \mathbb{Z}$ gives the wavelet coefficients. Once the coefficients d_l^j are computed, they remain unaltered in the subsequent calculations. This gives a very efficient algorithm **fast wavelet transform (FWT**, decomposition procedure) for the computation of wavelet coefficients. In matrix vector product form Eqs. (14) and (15) are written as

$$\mathbf{d} = \mathcal{W}\mathbf{c},$$

where $\mathbf{c} = \mathbf{c^J} = [c_0^J, c_1^J, \cdots, c_{2^J-1}^J]^T$ and $\mathbf{d} = [\mathbf{c^{J_0}}, \mathbf{d^{J_0}}, \mathbf{d^{J_0+1}}, \cdots, \mathbf{d^{J-1}}]^T, \mathbf{d^J} = [d_0^J, d_1^J, \cdots, d_{2^J-1}^J]^T$. The **inverse fast wavelet transform (IFWT**, reconstruction procedure) can be obtained in a similar manner.

The MATLAB function **[d] = fwt(x,D,λ)** gives the fast discrete periodized wavelet transform, where x is any vector of periodic function values, D ia wavelet genus and $\lambda = J - J_0$ is the depth of transform.

E. Connection coefficient

Any numerical scheme for solving differential equations must adequately represent the derivatives and non-linearities of the unknown function. In the case of wavelet bases, these approximations give rise to certain L_2 inner products of the basis functions, their derivatives and their translates, called the connection coefficients. In Fourier-based methods, since the products of the basis elements are also basis elements, the procedure does not face any difficulty. The numerical approximation of the connection coefficients which appear with the wavelet bases is unstable since the integrands are highly oscillatory. Scaling functions and wavelets do not have explicit analytical expressions but are implicitly determined by the two scale relations (5) and (6), it is necessary to develop algorithms to compute several connection coefficients, which occur in the application of the wavelet-Galerkin to differential equations. Specific algorithms have been devised by Latto et al. [50]. In the most general case we allow ϕ_l to be differentiated which gives rise to the n-term connection coefficients:

$$\wedge(l_1, l_2, \cdots, l_n, d_1, d_2, \cdots, d_n) = \wedge_{l_1 l_2 \cdots l_n}^{d_1 d_2 \cdots d_n} = \int_{-\infty}^{\infty} \prod_{i=1}^{n} \phi_{l_i}^{d_i}(x).$$

We can alter a doubly subscripted connection coefficient in to a singly subscripted one, and a triply subscripted connection coefficient in to a doubly subscripted one. We therefore define the two and three term connection coefficients as

$$\wedge_l^{d_1 d_2} = \int_{-\infty}^{\infty} \phi^{d_1}(x)\phi_l^{d_2}(x)dx, \tag{16}$$

TABLE I: $\phi, \psi \in C^{\alpha}(\mathbb{R}), H^{\beta}(\mathbb{R})$.

D/M	2/1	4/2	6/3	8/4	10/5	12/6	14/7	16/8	18/9	20/10
α	-	0	1	1	1	1	2	2	2	2
β	0	0	1	1	2	2	2	2	3	3

and

$$\wedge_{lm}^{d_1 d_2 d_3} = \int_{-\infty}^{\infty} \phi^{d_1}(x)\phi_l^{d_2}(x)\phi_m^{d_3}(x)dx, \tag{17}$$

where $d_i \geq 0$. The question of differentiability of ϕ and ψ is not fully understood (see [34] for details). Let space $C^{\alpha}(\mathbb{R})$ denotes the space of functions having continuous derivatives of order$\leq \alpha$ and $H^{\beta}(\mathbb{R}) = \{f \in L_2(\mathbb{R}) : f^d \in L_2(\mathbb{R}), |d| \leq \beta\}$. The regularity of Daubechies compactly supported scaling and wavelet functions is shown in Table I. We observe from Table I that same regularity could be achieved with less number of vanishing moments (e.g. for $D/M = 6/3$, $\phi, \psi \in C^1(\mathbb{R})$ and for $D/M = 12/6$ also $\phi, \psi \in C^1(\mathbb{R})$). So there are two different issues of maximum regularity and maximum number of vanishing moment (M) for $\psi(x)$ which should be chosen according to the application [34].

Let $f \in \mathcal{V}^{j,1} \cap C^d(\mathbb{R})$ then

$$f^{(d)}(x) = \sum_{l=-\infty}^{\infty} c_l^j \tilde{\phi}_l^{j,(d)}(x), \quad x \in \mathbb{R}, \tag{18}$$

$f^{(d)}$ will in general not belong to $\mathcal{V}^{j,1}$ so we project $f^{(d)}$ back onto $\mathcal{V}^{j,1}$

$$P_{\mathcal{V}^{j,1}} f^{(d)}(x) = \sum_{k=-\infty}^{\infty} c_k^{j(d)} \tilde{\phi}_k^j, \quad x \in \mathbb{R}, \tag{19}$$

$$c_k^{j(d)} = \int_{-\infty}^{\infty} f^{(d)} \tilde{\phi}_k^j(x)dx. \tag{20}$$

Then substituting (18) into (20) systems of equation can be represented in matrix-vector form

$$c^{j(d)} = \mathcal{D}^{(d)} c^j. \tag{21}$$

We will refer to the matrix $\mathcal{D}^{(d)}$ as the differentiation matrix of order d. Derivation of Eq. (21) is given in [51] where

$$[\mathcal{D}^{(d)}]_{k,<n+k>_{2^j}} = 2^{jd} \wedge_n^d, \quad k = 0, 1, \cdots, 2^j - 1,$$
$$n = 2 - D, 3 - D, \cdots, D - 2.$$

If the function to be differentiated is periodic with period p then we have

$$c^{j(d)} = D_1^{(d)} c^j,$$

where $\mathcal{D}_1^{(d)} = \frac{\mathcal{D}^{(d)}}{p^d}$ and $c^{j(d)} = [c_0^{j(d)}, c_1^{j(d)}, \cdots, c_{2^j-1}^{j(d)}]$.

Consider the $f \in C^M(\mathbb{R})$, then the approximation error will be

$$e^{j,1}(x) = f(x) - (P_{\mathcal{V}^{j,1}} f)(x) \tag{22}$$

and $||e^{j,1}||_{\infty} = O(2^{-jM})$ (this is exponential decay with respect to j. Furthermore, the greater the number of vanishing moments M, the faster the decay. In contrast, FDM and FEM yields convergence rates that are only algebraic in N (degree of freedom), typically $O(N^{-2})$ or $O(N^{-4})$), where $P_{\mathcal{V}^{j,1}} f(x)$ is orthogonal projection of f on $\mathcal{V}^{j,1}$ as defined in Sec. II C. It is comparable to accuracy of spectral method :for problems with smooth solutions convergence rates are $O(e^{-cN})$ or $O(e^{-c\sqrt{N}})$

Furthermore, the space $\mathcal{V}^{j,1}$ can exactly represent polynomials up to degree $M - 1$ (error term in Eq. (22) will be zero), however differentiation matrix is accurate of order $2M$. This doubling of the accuracy is also encountered in FEM and is known as **superconvergence** (which is lost in case of general boundary condition).

III. WAVELET BASED NUMERICAL METHODS

A. Wavelet Galerkin method

In Galerkin method the degrees of freedom are the expansion coefficients of a set of basis functions and these expansion coefficients are not in physical space (means in wavelet space). Moreover, in wavelet Galerkin methods the treatment of nonlinearities is complicated which can be handled with couple of techniques.

- Using the connection coefficients discussed in Sec. II E (expensive approach due to the summation over multiple indices).
- Using the quadrature formula [52] (loses its accuracy due to approximate calculation).
- Pseudo approach [23] (first map wavelet space to physical space, compute nonlinear term in physical space and then back to wavelet space, this approach is not very practical because it requires transformation between the physical space and wavelet space).

The derivatives can be obtained using the differential matrix discussed in Sec. II E.

B. Wavelet collocation method

Collocation method involve numerical operators acting on point values (collocation points) in the physical space. Generally, wavelet collocation methods are created by choosing a wavelet and some kind of grid structure which will be computationally adapted. In effect, one obtains finite differences on nonuniform grid. The treatment of nonlinearities in wavelet collocation method is straightforward task due to collocation nature of algorithm. The derivatives can be obtained like interpolation method for projecting f on $\mathcal{V}^j$ discussed in Sec. II C 2.

Moreover, proofs are easier with Galerkin methods, whereas implementation is more practical with collocation methods.

C. Numerical example

We illustrate various wavelet based numerical methods for heat equation in one–dimension with periodic boundary conditions.

$$\begin{aligned} u_t &= \nu u_{xx} + f(x), t > 0 \\ u(x,0) &= h(x), \quad 0 \le x \le 1, \end{aligned} \tag{23}$$

where ν is a positive constant, $f(x) = f(x+1)$ and $h(x) = h(x+1)$.

1. *Method based on scaling function expansion*

Let us first leave the spatial variable x continuous and discretize only the time to obtain the Euler scheme:

$$u_t^n = \frac{u^{n+1} - u^n}{\delta t} = \nu u_{xx}^n + f(x). \tag{24}$$

Now to obtain solution $u_j \in \mathcal{V}^{j,1}$ we apply **wavelet-Galerkin method** to (24) with approximation of the form

$$u^j(x) = \sum_{k=0}^{2^j-1} c_k^j \tilde{\phi}_k^j(x), \tag{25}$$

where c_k^j is the unknown coefficient of scaling function expansion of $u^j(x)$. The Galerkin discretization scheme gives

$$\left(c_u^{n+1} - c_u^n\right) = \nu \delta t \mathcal{D}^{(2)} c_u^n + \delta t c_f, \tag{26}$$

where c_u denote the vector of scaling function coefficients corresponding to u and c_f is given by

$$(c_f)_l^j = \int_0^1 f(x)\tilde{\phi}_l^j(x)dx,$$

and $\mathcal{D}^{(2)}$ is discussed in Sec. II E. Finally, we arrive at the linear algebraic system

$$\mathcal{A}c_u = \mathcal{F}, \tag{27}$$

which can be solved using any linear algebraic solver. Hence this comes under the category 1 discussed in Sec. I.

2. Method based on wavelet function expansion

By rewriting Eq. (27) in the following form

$$\mathcal{W}\mathcal{A}\mathcal{W}^T\mathcal{W}c_u = \mathcal{W}\mathcal{F}, \tag{28}$$

and substituting $d_u = \mathcal{W}c_u$ and $d_f = \mathcal{W}c_f$ in Eq. (28)

$$\mathcal{W}\mathcal{A}\mathcal{W}^T d_u = d_{\mathcal{F}}. \tag{29}$$

This comes under the category 2 discussed in Sec. I.

3. Method based on physical space representation

Multiplying Eq. (26) by T^j

$$T^j\left(c_u^{n+1} - c_u^n\right) = \nu\delta t T^j\mathcal{D}^{(2)}c_u^n + \delta t\mathcal{T}^j c_f, \tag{30}$$

$$\left(u^{n+1} - u^n\right) = \nu\delta t T^j\mathcal{D}^{(2)}c_u^n + \delta t f. \tag{31}$$

By redefining $f = \{f(x_l)\}_{l=0}^{2^J-1}$, this can also seen as a **wavelet collocation method** for the solution of (23). This is also based on scaling function expansion and does not exploit any wavelet compression, hence it also comes under the category 1 discussed in Sec. I.

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